# EVEN-PANCYCLIC SUBGRAPHS OF MESHES

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Abstract. A graph $G$ of order $n$ is *even-pancyclic* if it contains cycles of all possible even lengths $4, 6, 8, \ldots, 2\left\lfloor \frac{n}{2} \right\rfloor$. The 2-dimensional mesh $M(m,n)$ is the Cartesian product of the two paths $P_m$ and $P_n$. We present several results on even-pancyclic subgraphs of meshes.

## 1. Introduction

We follow the notation of [1]. The reader is reminded that a graph $G$ of order $n$ is *pancyclic* if it contains cycles of all lengths $k$, $3 \leq k \leq n$. Pancyclic graphs have been studied in [2, 3, 4, 5]. A graph $G$ of order $n$ is *even-pancyclic* if it contains cycles of all even lengths $2k$, $2 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$. The *2-dimensional mesh*, or *2-mesh*, $M(m,n)$ is the Cartesian product of paths $P_m$ and $P_n$. The *n-dimensional mesh* $M(a_1, a_2, \ldots, a_n)$ is the Cartesian product of paths of orders $a_1, a_2, \ldots, a_n$. A mesh is *even* if its order is even, otherwise it is *odd*. The *ladder* $L_n$ is the mesh $M(2,n)$. The ladder $M(2,n)$ contains $n$ *rungs* corresponding to the $n$ copies of $P_2$. A graph $G$

is *traceable* if it contains a Hamiltonian path. The *fan* $F_n$, $n \geq 3$, is $K_1$ joined with the path $P_{n-1}$. For $n \geq 3$, the *wheel* $W_{1,n}$ is $K_1$ joined with the cycle $C_n$. Clearly, the fan $F_n$ and the wheel $W_{1,n}$ are pancyclic and the ladder $L_n$ is even-pancyclic. In fact we can say the following:

**Theorem 1.** *Let $G$ be traceable, then $G \times K_2$ is even-pancyclic.*

*Proof.* Since $G$ has a Hamiltonian path, $G \times K_2$ contains the ladder $L_n$ where $n$ is the order of $G$. □

**Theorem 2.** *Let $G$ be traceable, $K_1 + G$ is pancyclic.*

*Proof.* If $G$ is traceable, then $K_1 + G$ contains the fan $F_{n+1}$ as a subgraph where $n$ is the order of $G$. □

The *generalized wheel* $W_{m,n}$ defined by Buckley and Harary [6] is $\overline{K_m} + C_n$.

**Theorem 3.** *The generalized wheel $W_{m,n}$ is pancyclic if and only if $m \leq n$.*

*Proof.* Label the nodes of $\overline{K_m}$ as $u_1, u_2, \ldots, u_m$ and the nodes of $C_n$ as $v_1, v_2, \ldots, v_n$. Clearly, $W_{m,n}$ contains $W_{1,n}$ as a subgraph. In fact, the induced subgraph on $u_1, v_1, v_2, \ldots, v_n$ is $W_{1,n}$ which is pancyclic containing cycles $C_3, C_4, \ldots, C_{n+1}$. There are $n$ identical Hamilton cycles $C_{n+1}$ up to labeling, one such cycle is $u_1v_1, v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nu_1$. Each edge $v_{i-1}v_i$, $i = 2, \ldots, n$, is an edge from the original cycle $C_n$ and not an edge from the join, whereas $u_1v_1$ and $v_nu_1$ are edges resulting from the join operation. To connect the remaining $m-1$ nodes $u_2, u_3, \ldots, u_m$ of $\overline{K_m}$ and obtain cycles $C_{n+2}, C_{n+3}, \ldots, C_{n+m}$ simply replace one of the $v_{i-1}v_i$ edges with two edges $v_{i-1}u_j$ and $u_jv_i$, $j = 2, \ldots, m$. This can be done at most $n-1$ times. Thus, $m$ can be at most $n$. □

A *ruler* is a strictly increasing finite sequence of nonnegative integers called *marks*. A *segment* of a ruler is the space between two adjacent marks. The number of segments is the number of marks less one. A ruler is *complete* if the set of all distances it can measure is $\{1, 2, 3, ..., k\}$ for some integer $k \geq 1$. By convention, the first mark is 0. A complete ruler can therefore measure all distances from 1 to its length. A ruler is *perfect* if it is complete and no complete ruler with the same length possesses fewer marks. [7]

## 2. Even-Pancyclic Subgraphs of Ladders

We wish to determine the minimum number of edges in even-pancyclic spanning subgraphs of ladders. The ladders $L_2$ and $L_3$ have the property that all edges are required in order for them to be even-pancyclic, while $L_4$ only requires one of its interior rungs. (See Figure 1)

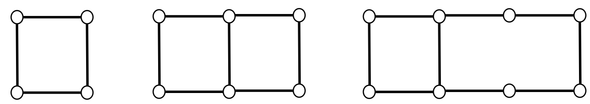


FIGURE 1

If we label the rungs of the ladder $L_n$ as $0, 1, 2, \ldots, n-1$, the numbers that correspond to the rungs necessary for a subgraph of $L_n$ with $1 \leq n \leq 8$ to be even-pancyclic are as follows:

| $L_n$ | Rungs required |
|---|---|
| $L_1$ | $\{0\}$ |
| $L_2$ | $\{0,1\}$ |
| $L_3$ | $\{0,1,2\}$ |
| $L_4$ | $\{0,1,3\}$ or $\{0,2,3\}$ |
| $L_5$ | $\{0,1,3,4\}$ , $\{0,1,2,4\}$ , or $\{0,2,3,4\}$ |
| $L_6$ | $\{0,1,3,5\}$, $\{0,1,2,5\}$, $\{0,3,4,5\}$, or $\{0,2,4,5\}$ |
| $L_7$ | $\{0,1,4,6\}$, or $\{0,2,5,6\}$ |
| $L_8$ | there are 12 of them |

These are exactly the marks required for a ruler to be of perfect. In fact, there is a direct correspondence between the marks of a perfect ruler and the rungs in an edge minimal even-pancyclic spanning subgraph of a ladder. Recall, that the number of marks is one more than the number of segments in the ruler. The number of segments in a perfect ruler with length $k$ is given in the sequence A103298 of The On-Line Encyclopedia of Integer Sequences (OEIS) [8]. The minimum number of rungs is, therefore, one more than this number. This gives us the following theorem.

**Theorem 4.** *The minimum number of edges in an even-pancyclic spanning subgraph of the ladder $L_n$ is $2n + a_{n-1} - 1$, where $a_i$ is the number of segments in a perfect ruler with length $i$ given by the sequence A103298 of the OEIS.*

*Proof.* The rails of the ladder are the two copies of $P_n$ which have $n-1$ edges. The number of rungs needed is equal to the number of marks on a perfect ruler of length $n-1$, which is $a_{n-1}+1$. The constructed subgraph has $2(n-1)+a_{n-1}+1 = 2n+a_{n-1}-1$ edges. This number is minimal because if we remove an edge from one of the rails we lose the Hamiltonian cycle, if we remove a rung we would have a perfect ruler with less marks. By definition, the perfect ruler has the fewest marks. To see that this subgraph is even-pancyclic, note that all lengths from 1 to $n-1$ are represented along the rails between at least two of the rungs, this yields all cycles $C_4, C_6, \ldots, C_{2n}$. □

*Remark* 1. The number of edge minimal even-pancyclic spanning subgraphs of the ladder $L_{n+1}$ is the same as the number of perfect rulers of length $n$ and this is given by the sequence A103300 of the OEIS.

*Remark* 2. The total number of even-pancyclic spanning subgraphs of the ladder $L_{n+1}$ is the same as the number of complete rulers of length $n$ and this is given by the sequence A103295 of the OEIS.

As for *uniquely* even-pancyclic subgraphs of ladders, these correspond to perfect Golomb rulers. A *Golomb ruler* is a ruler with the property that no two pairs of marks are the same distance apart. A perfect Golomb ruler measures all of the distances up to its length uniquely. Golomb [9] showed there are no perfect rulers with more than 4 marks. With this in mind, the only uniquely even-pancyclic subgraphs of ladders are shown in Figure 2.

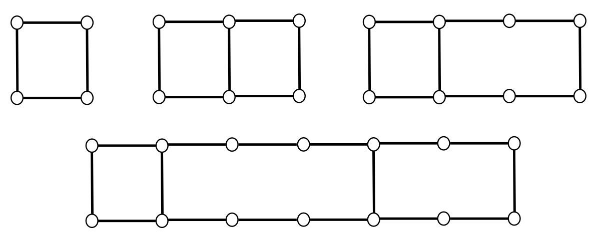

FIGURE 2. The Uniquely Even-Pancyclic Subgraphs of Ladders

## 3. EVEN-PANCYCLIC SUBGRAPHS OF MESHES

We wish to find even-pancyclic spanning subgraphs of higher dimensional meshes. For 2-dimensional meshes, we present the following theorem.

**Theorem 5.** *Every 2-dimensional mesh $M(m,n)$ contains even-pancyclic subgraph on $mn + a_{\lfloor \frac{mn}{2} \rfloor - 1} - mn \mod 2 - 1$ edges, where $a_i$ is the number of segments of a perfect ruler with length $i$ given by the sequence A103298 of the OEIS.*

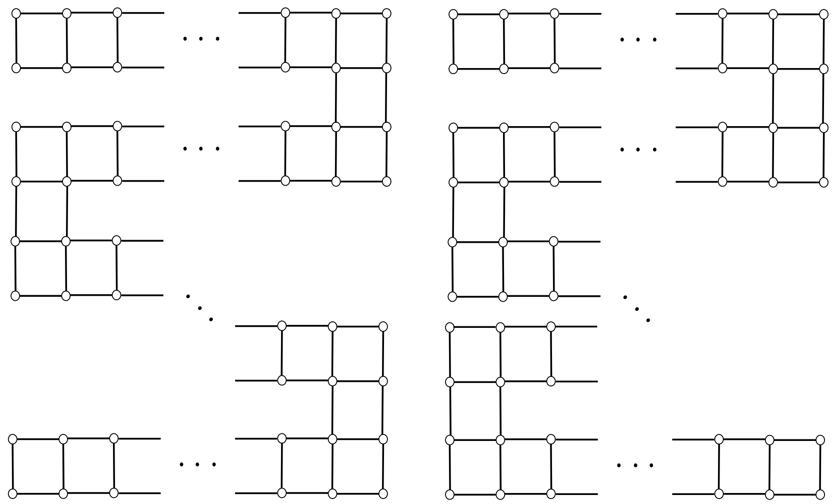


FIGURE 3. The "twisted" ladder $L_{kn}$

*Proof.* We divide into two cases, even and odd meshes, and show how such a subgraph is constructed.

*Case 1.* The mesh is even. A mesh is even if its order is even. For a 2-dimensional mesh $M(m,n)$, this means that at least one of $m$ and $n$ is even. Without loss of generality, say $m$ is even, hence $m = 2k$. As shown in Figure 3, an even mesh contains a subgraph that represents a "twisted" ladder. The difference depends on whether $k$ is even or $k$ is odd. In either case, number the rungs from 0 to $\lfloor \frac{mn}{2} \rfloor - 1 = kn - 1$, then remove all rungs except those corresponding to the marks of a perfect ruler of length $kn-1$. The resulting subgraph is even-pancyclic. Thus, every even 2-dimensional mesh $M(m,n)$ contains a subgraph that is even-pancyclic and whose total number of edges is $mn + a_{\left(\frac{mn}{2}-1\right)} - 1$, where $a_i$ is the number of segments of a perfect ruler with length $i$ given by the sequence A103298 of the OEIS.

*Case 2.* The mesh is odd. If $M(m,n)$ is odd, then $m$ and $n$ are both odd. This means that $m$ and $n$ are either of the form $4t+1$ or $4t+3$. We again divide into two cases.

*Case 2a.* At least one of $m$ or $n$ is of the form $4t+1$. In this case, label a corner node, one of degree 2, $v$. The graph $M(m,n) - v$ contains the spanning "twisted" ladder subgraph shown in Figure 4.

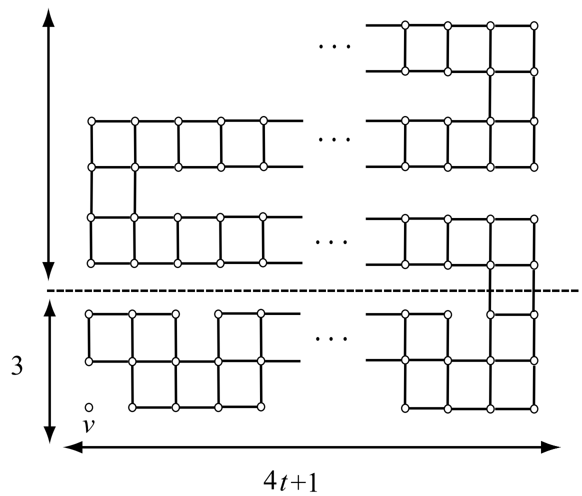


FIGURE 4

If we again number the rungs from 0 to $\lfloor \frac{mn}{2} \rfloor - 1$ and remove all rungs except those corresponding to the marks of a perfect ruler of length $\lfloor \frac{mn}{2} \rfloor - 1$, we will obtain the required even-pancyclic subgraph.

*Case 2b.* Both $m$ and $n$ are of the form $4t + 3$.

In this case, label a node of degree 3 a distance 2 away from a corner node, $v$. The graph $M(m, n) - v$ contains the spanning "twisted" ladder subgraph shown in Figure 5.

Again number the rungs from 0 to $\lfloor \frac{mn}{2} \rfloor - 1$ and remove all rungs except those corresponding to the marks of a perfect ruler of length $\lfloor \frac{mn}{2} \rfloor - 1$. If we do this, we will obtain an even-pancyclic subgraph on $mn + a_{\frac{(mn-1)}{2} - 1} - 2$ edges, where $a_i$ is the number of segments of a perfect ruler with length $i$ given by the sequence A103298 of the OEIS. □

Since every higher dimensional mesh contains a spanning 2-dimensional mesh, the corollary follows.

**Corollary 1.** *Every mesh of order $p$ contains even-pancyclic spanning subgraphs on $p + a_{\lfloor \frac{p}{2} \rfloor - 1} - p \mod 2 - 1$ edges, where $a_i$ is the number of segments of a perfect ruler with length $i$ given by the sequence A103298 of the OEIS.*

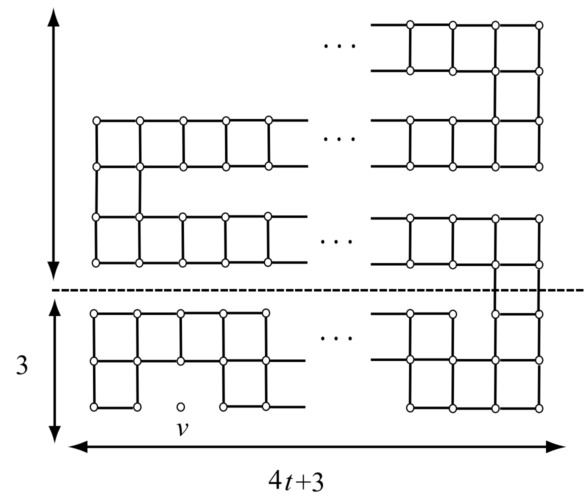


Figure 5

## 4. Pancyclic Subgraphs of Fans and Wheels

Consider the fan $F_n$ and the wheel graph $W_{1,n}$. Since the fan $F_{n+1}$ is a subgraph of the wheel $W_{1,n}$, any result regarding spanning subgraphs for the first will apply to the latter. Recall, the fan $F_n$, $n \geq 3$, is $K_1 + P_{n-1}$. The $K_1$ node is called the *core*. The edges incident with the core are called *spokes*. For the fan $F_n$, we can label the spokes $0, 1, 2, \ldots, n-2$. Then remove the spokes which do not correspond to the marks on a perfect ruler of length $n-2$. Therefore, we have the following:

**Theorem 6.** *The fan $F_n$ contains edge minimal pancyclic subgraphs on $n + a_{n-2} - 1$ edges.*

*Proof.* The number of edges in the path is $n-2$ to this we add only the spokes that correspond to the marks on a perfect ruler of length $n-2$ which is $a_{n-2}+1$ where $a_i$ is the number of segments of a perfect ruler with length $i$ given by the sequence A103298 of the OEIS. This number is minimal because if we remove an edge from the path we lose the Hamiltonian cycle and if we remove a spoke we would have a perfect ruler with less marks. By definition the perfect ruler has the fewest marks. To see that the subgraph is pancyclic, note that all distances from 1 to $n-2$ are represented between at least two spokes, together with the 2 spokes, yields all cycles $C_3, C_4, \ldots, C_n$. □

**Corollary 2.** *The wheel $W_{1,n}$ contains edge minimal pancyclic subgraphs on $n + a_{n-1}$ edges.*

*Proof.* $W_{1,n}$ contains the fan $F_{n+1}$ as a subgraph. □

*Remark* 3. The number of edge minimal pancyclic subgraphs of the fan $F_{n+2}$ is the same as the number of perfect rulers of length $n$ and this is given by the sequence A103300 of the OEIS. The wheel $W_{1,n+1}$ contains $n+1$ subgraphs isomorphic to $F_{n+2}$. Therefore, the number of edge minimal pancyclic subgraphs of the wheel $W_{1,n+1}$ is $(n+1)a_n$ where $a_n$ is the number of perfect rulers of length $n$ given by the sequence A103300 of the OEIS.

*Remark* 4. The total number of pancyclic subgraphs of the fan $F_{n+2}$ is the same as the number of complete rulers of length n and this is given by the sequence A103295 of the OEIS. Similarly, the total number of pancyclic subgraphs of the wheel $W_{1,n+1}$ is $n + 1$ times this number.

## References


[1] M. Lewinter and F. Buckley. *A Friendly Introduction to Graph Theory.* Prentice Hall, Upper Saddle River, New Jersey, 2003.

[2] J.A. Bondy, Pancyclic graphs, in: *Proceedings of the Second Louisiana Conference on Combinatorics, Graph Theory and Computing*, Louisiana State University, Baton Rouge, LA, (1971) 167-172.

[3] J.A. Bondy, Pancyclic graphs I, *J. Combin. Theory Ser B.*, **11** (1971) 80-84.

[4] Y.B. Shi, Some theorems of uniquely pancyclic graphs, *Discrete Math.*, **59** (1986) 167-180.

[5] B. Randerath, I. Schiermeyer, M. Tewes, L. Volkmann, Vertex pancyclic graphs, *Discrete Applied Mathematics,* **120** (2002) 219-237.

[6] F. Buckley and F. Harary, On the euclidean dimension of a wheel, *Graphs and Combinatorics,* **4** (1988) 23-30.

[7] P. Luschny, Perfect rulers, 2005, published electronically at: http://www.luschny.de/math/rulers/index.html.

[8] N. J. A. Sloane, Sequences A103295, A103298, and A103300 in The On-Line Encyclopedia of Integer Sequences, published electronically at: http://www.research.att.com/˜njas/sequences/index.html.

[9] S. W. Golomb, Numbering the Nodes of a Graph, *Computing and Graph Theory* (ed. R. C. Read), Academic Press, NY, (1972) 23-37.